Characterisation of smooth functions with given growth

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We consider the space $\mathbf{R}^m$ equipped with the standard norm $|\zeta|$ and the scalar product $\langle \zeta, \eta \rangle$ for $\zeta \in \mathbf{R}^m$ and $\eta \in \mathbf{R}^m$. We denote by $\mathbf{B}^m$ the unit ball in $\mathbf{R}^m$.

Let $\Omega \subseteq \mathbf{R}^m$ be a domain. For a differentiable mapping $f : \Omega \to \mathbf{R}^n$, denote by $Df(\zeta)$ its differential at $\zeta \in \Omega$, and by

$$\|Df(\zeta)\| = \sup_{\ell \in \partial \mathbf{B}^m} |Df(\zeta)\ell|$$

the norm of the linear operator $Df(\zeta) : \mathbf{R}^m \to \mathbf{R}^n$.

Our results are mainly motivated by the following surprising theorem of Pavlović:

A continuously differentiable complex-valued function $f(\zeta)$ in the unit ball $\mathbf{B}^m$ is a Bloch function, i.e.,

$$\sup_{\zeta \in \mathbf{B}^m} (1 - |\zeta|^2)\|Df(\zeta)\|$$

is finite, if and only if the following quantity if finite:

$$\sup_{\zeta,\, \eta \in \mathbf{B}^m,\, \zeta \neq \eta} \sqrt{1 - |\zeta|^2}\sqrt{1 - |\eta|^2}\frac{|f(\zeta) - f(\eta)|}{|\zeta - \eta|}.$$

Moreover, these numbers are equal.

The above result appeared in

M. Pavlović, *On the Holland–Walsh characterization of Bloch functions*, Proc. Edinburgh Math. Soc. **51** (2008), 439–441.

The results that will be presented are given in the author recent work (which is motivated by the previously mentioned Pavlović's work)

M. Marković, *Differential-free characterisation of smooth functions with controlled growth*, Canadian Mathematical Bulletin, to appear.

This paper contains some generalizations and improvements of the Pavlović result on the Holland-Walsh type characterization of the Bloch space of continuously differentiable (smooth) functions in the unit ball in $\mathbf{R}^m$.

As Pavlović observed, his result is actually two-dimensional. Namely, if one proves it for continuously differentiable functions $\mathbf{B}^2 \to \mathbf{C}$, then the general case (the case of continuously differentiable functions $\mathbf{B}^m \to \mathbf{C}$) follows from it.

We will derive it using our main result.

Since for an analytic function $f(z)$ in the unit disc $\mathbf{B}^2$ we have

$$\|Df(z)\| = |f'(z)|$$

for every $z \in \mathbf{B}^2$, the first part of the Pavlović result (without the equality statement) is the Holland–Walsh characterization of analytic functions in the Bloch space in the unit disc. This is Theorem 3 in their work

F. Holland and D. Walsh, *Criteria for membership of Bloch space and its subspace, BMOA*, Math. Ann. **273** (1986), 317–335,

which says that $f(z)$ is a Bloch function if and only if

$$\sqrt{1-|z|^2}\sqrt{1-|w|^2}\frac{|f(z)-f(w)|}{|z-w|}$$

is bounded as a function of two variables $z \in \mathbf{B}^2$ and $w \in \mathbf{B}^2$ for $z \neq w$.

This characterisation of analytic Bloch functions in the unit ball is given by Ren and Tu in

G. Ren and C. Tu, *Bloch space in the unit ball of* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **133** (2005), 719–726.

Our aim here is to obtain a characterisation result (which resembles the Pavlović result) of continuously differentiable mappings that satisfy a certain growth condition.

We need to introduce some notation.

Let $\mathbf{w}(\zeta)$ be an everywhere positive continuous function in a domain $\Omega \subseteq \mathbf{R}^m$ (a weight function in Ω). We will consider continuously differentiable mappings in Ω that map this domain into $\mathbf{R}^n$ and satisfy the following growth condition

$$\|f\|_{\mathbf{w}}^{\mathbf{b}} := \sup_{\zeta \in \Omega} \mathbf{w}(\zeta)\|Df(\zeta)\| < \infty.$$

We say that $\|f\|_{\mathbf{w}}^{\mathbf{b}}$ is the $\mathbf{w}$-Bloch semi-norm of the mapping f (it is easy to check that it has indeed all semi-norm properties).

We denote by $\mathcal{B}_{\mathbf{w}}$ the space of all continuously differentiable mappings $f : \Omega \to \mathbf{R}^n$ with the finite $\mathbf{w}$-Bloch semi-norm. The space $\mathcal{B}_{\mathbf{w}}$ we call $\mathbf{w}$-Bloch space.

If $\Omega = \mathbf{B}^m$ and $\mathbf{w}(\zeta) = 1 - |\zeta|^2$ for $\zeta \in \mathbf{B}^m$, we just say the Bloch space, and denote it by $\mathcal{B}$.

One of our aims is to give a differential-free description of the **w**-Bloch space and a differential-free expression for **w**-Bloch semi-norm.

In order to do that, for a given weight $\mathbf{w}(\zeta)$ in a domain Ω, we now introduce a new everywhere positive function $\mathbf{W}(\zeta,\eta)$ on the product domain $\Omega\times\Omega$ that satisfies the following four conditions.

For every $\zeta\in\Omega$ and $\eta\in\Omega$,

$$\begin{aligned} &(W_1) \quad \mathbf{W}(\zeta,\eta)=\mathbf{W}(\eta,\zeta);\\ &(W_2) \quad \mathbf{W}(\zeta,\zeta)=\mathbf{w}(\zeta);\\ &(W_3) \quad \liminf_{\eta\to\zeta}\mathbf{W}(\zeta,\eta)\ge\mathbf{W}(\zeta,\zeta)=\mathbf{w}(\zeta);\\ &(W_4) \quad d_{\mathbf{w}}(\zeta,\eta)\mathbf{W}(\zeta,\eta)\le|\zeta-\eta|, \end{aligned}$$

where $d_{\mathbf{w}}(\zeta,\eta)$ is the **w**-distance between $\zeta\in\Omega$ and $\eta\in\Omega$, which is obtained in the following way:

$$d_{\mathbf{w}}(\zeta,\eta)=\inf_{\gamma}\int_{\gamma}\frac{|d\omega|}{\mathbf{w}(\omega)},$$

where the infimun is taken over all piecewise smooth curves $\gamma\subseteq\Omega$ connecting ζ and η (it is well known that $d_{\mathbf{w}}(\zeta,\eta)$ is a distance function in the domain Ω). We say that $\mathbf{W}(\zeta,\eta)$ is admissible for $\mathbf{w}(\zeta)$.

Of course, one can pose the existence question concerning $\mathbf{W}(\zeta,\eta)$ if $\mathbf{w}(\zeta)$ is given.

We will prove that the following functions $\mathbf{W}(\zeta,\eta)$ are admissible for the given functions $\mathbf{w}(\zeta)$.

1. The function

$$\mathbf{W}(\zeta,\eta) = \begin{cases} \mathbf{w}(\zeta), & \text{if } \zeta = \eta, \\ |\zeta-\eta|/d_{\mathbf{w}}(\zeta,\eta), & \text{if } \zeta \neq \eta. \end{cases}$$

   in $\Omega \times \Omega$ is admissible for any given $\mathbf{w}(\zeta)$ in Ω.
2. If $\mathbf{w}(\zeta) = 1 - |\zeta|^2$ for $\zeta \in \mathbf{B}^m$, then $d_{\mathbf{w}}(\zeta,\eta)$ is the hyperbolic distance in the unit ball $\mathbf{B}^m$. One of the admissible functions is

$$\mathbf{W}(\zeta,\eta) = \sqrt{1-|\zeta|^2}\sqrt{1-|\eta|^2}.$$

   From this fact we deduce the Pavlović result stated at the beginning.
3. If Ω is a convex domain and if $\mathbf{w}(\zeta)$ is a decreasing function in $|\zeta|$, then

$$\mathbf{W}(\zeta,\eta) = \min\{\mathbf{w}(\zeta), \mathbf{w}(\eta)\}$$

   is admissible for $\mathbf{w}(\zeta)$. It would be of interest to find such simple admissible functions for more general domains Ω and/or more general functions $\mathbf{w}$.

For a mapping $f : \Omega \to \mathbf{R}^n$ introduce now the quantity

$$\|f\|_{\mathbf{W}}^{\mathbf{l}} \quad := \sup_{\zeta,\,\eta\in\Omega,\,\zeta\neq\eta} \mathbf{W}(\zeta,\eta)\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|}.$$

We call it the **W**-Lipschitz semi-norm (it is also an easy task to check that it is indeed a semi-norm).

The space of all continuously differentiable mappings $f : \Omega \to \mathbf{R}^n$ for which its **W**-Lipschitz semi-norm $\|f\|_{\mathbf{W}}^{\mathbf{l}}$ is finite is denoted by $\mathcal{L}_{\mathbf{W}}$.

Note that if $\mathbf{W}(\zeta,\eta)$ is not symmetric, we can replace it by

$$\tilde{\mathbf{W}}(\zeta,\eta) = \max\{\mathbf{W}(\zeta,\eta), \mathbf{W}(\eta,\zeta)\}$$

which produces the same Lipschitz type semi-norm.

Our main result in the paper shows that for any continuously differentiable mapping $f : \Omega \to \mathbf{R}^n$ we have

$$\|f\|_{\mathbf{w}}^{\mathbf{b}} = \|f\|_{\mathbf{W}}^{\mathbf{l}};$$

i.e., the **w**-Bloch semi-norm is equal to the **W**-Lipschitz semi-norm of the mapping f.

As a consequence we have the coincidence of the two spaces $\mathcal{B}_{\mathbf{w}} = \mathcal{L}_{\mathbf{W}}$.

Thus, the space $\mathcal{B}_{\mathbf{w}}$ may be described as

$$\mathcal{B}_{\mathbf{w}} = \left\{ f : \Omega \to \mathbf{R}^n : \sup_{\zeta,\, \eta \in \Omega,\, \zeta \neq \eta} \mathbf{W}(\zeta, \eta) \frac{|f(\zeta) - f(\eta)|}{|\zeta - \eta|} < \infty \right\},$$

where $\mathbf{W}(\zeta, \eta)$ is any admissible function for $\mathbf{w}(\zeta)$.

This is the content of the following theorem.

Theorem

Let $\Omega \subseteq \mathbf{R}^m$ be a domain and let $f : \Omega \to \mathbf{R}^n$ be a continuously differentiable mapping. Let $\mathbf{w}(\zeta)$ be positive and continuous in Ω, and let $\mathbf{W}(\zeta, \eta)$ be an admissible function for $\mathbf{w}(\zeta)$. If one of the numbers $\|f\|^{\mathbf{b}}_{\mathbf{w}}$ and $\|f\|^{\mathbf{l}}_{\mathbf{W}}$ is finite, then both numbers are finite and equal.

We will remark the following fact. Let $\mathbf{w}(\zeta)$ be a weight in a domain $\Omega \subseteq \mathbf{R}^m$. Observe that we have

$$\sup_{\zeta \in \Omega} \mathbf{w}(\zeta) = \sup_{\zeta,\, \eta \in \Omega,\, \zeta \neq \eta} \mathbf{W}(\zeta, \eta),$$

where $\mathbf{W}(\zeta, \eta)$ is admissible for $\mathbf{w}(\zeta)$. This remark is a direct consequence of the fact that we can set the identity $f(\zeta) = \mathrm{Id}(\zeta)$ in our theorem.

We will now discus the Pavlović result.

As we have already said, if we take

$$\mathbf{w}(\zeta) = 1 - |\zeta|^2, \quad \zeta \in \mathbf{B}^m,$$

then $\mathbf{w}$-distance is the hyperbolic distance - for the hyperbolic distance between $\zeta \in \mathbf{B}^m$ and $\eta \in \mathbf{B}^m$ we will use the usual notation $\rho(\zeta, \eta)$ (instead of $d_{\mathbf{w}}(\zeta, \eta)$).

One more expression for the hyperbolic distance in the unit ball is given by

$$\sinh^2 \rho(\zeta, \eta) = \frac{|\zeta - \eta|^2}{(1 - |\zeta|^2)(1 - |\eta|^2)}$$

(see the book of Vuorinen).

Using the elementary inequality

$$t \leq \sinh t,$$

one deduces that

$$\mathbf{W}(\zeta, \eta) = \sqrt{1 - |\zeta|^2}\sqrt{1 - |\eta|^2}$$

has W_4-property, and therefore it is admissible for $\mathbf{w}(\zeta) = 1 - |\zeta|^2$.

The Pavlović result now follows.

We will mention now some other consequences of our main result.

Corollary

Let $\mathbf{w}(\zeta)$ be an everywhere positive, continuous and decreasing function of $|\zeta|$ in a convex domain $\Omega \subseteq \mathbf{R}^m$. Then we have

$$\sup_{\zeta\in\Omega} \mathbf{w}(\zeta)\|Df(\zeta)\| \quad = \sup_{\zeta,\,\eta\in\Omega,\,\zeta\neq\eta} \min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\}\frac{|f(\zeta)-f(\eta)|}{|\zeta-\eta|}$$

for every continuously differentiable mapping $f:\Omega\to\mathbf{R}^n$.

Let

$$\mathbf{W}(\zeta,\eta)=\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\},$$

for $(\zeta,\eta)\in\Omega\times\Omega$. We have only to check if $\mathbf{W}(\zeta,\eta)$ satisfies conditions $(W_1)-(W_4)$ and to apply our main theorem.

It is clear that $\mathbf{W}(\zeta,\eta)$ is symmetric, and that $\mathbf{W}(\zeta,\zeta)=\mathbf{w}(\zeta)$. Since $\mathbf{W}(\zeta,\eta)$ is continuous in $\Omega\times\Omega$, the (W_3)-condition for $\mathbf{W}(\zeta,\eta)$ obviously holds.

Therefore, it remains to check if the following inequality is true:

$$d_{\mathbf{w}}(\zeta,\eta)\min\{\mathbf{w}(\zeta),\mathbf{w}(\eta)\}\le|\zeta-\eta|,\quad (\zeta,\eta)\in\Omega\times\Omega.$$

Let $\zeta \in \Omega$ and $\eta \in \Omega$ be arbitrary and fixed and let $\gamma \subseteq \Omega$ be among piecewise smooth curves that joint ζ and η.

We have

$$\begin{aligned} d_{\mathbf{w}}(\zeta, \eta) = \inf_{\gamma} \int_{\gamma} \frac{|d\omega|}{\mathbf{w}(\omega)} &\leq \int_{[\zeta,\eta]} \frac{|d\omega|}{\mathbf{w}(\omega)} \\ &\leq \int_{[\zeta,\eta]} \max_{\omega \in [\zeta,\eta]} \left\{ \frac{1}{\mathbf{w}(\omega)} \right\} |d\omega| \\ &\leq \max \left\{ \frac{1}{\mathbf{w}(\zeta)}, \frac{1}{\mathbf{w}(\eta)} \right\} \int_{[\zeta,\eta]} |d\omega| \\ &= \max \left\{ \frac{1}{\mathbf{w}(\zeta)}, \frac{1}{\mathbf{w}(\eta)} \right\} |\zeta - \eta| \\ &= \min\{\mathbf{w}(\zeta), \mathbf{w}(\eta)\}^{-1} |\zeta - \eta|, \end{aligned}$$

where we have used in the fourth step our assumption that $\mathbf{w}(\omega)$ is decreasing in $|\omega|$ and that the maximum modulus of points on a line segment is attained at an endpoint.

The inequality we need follows.

In light of the above corollary we will consider now the Pavlović result.

Since the function

$$\mathbf{w}(\zeta) = 1 - |\zeta|^2$$

is decreasing in $|\zeta|$ in the unit ball $\mathbf{B}^m$, the above corollary produces a new Holland-Walsh type characterisation of continuously differentiable Bloch mappings.

However, notice that

$$\min\{A, B\} \leq \sqrt{A}\sqrt{B}$$

for all non-negative numbers A and B.

Because of this inequality, it seems that Corollary 2 improves the Pavlović result stated at the beginning.

Here is the next corollary

Corollary

Let $\mathbf{w}(\zeta)$ *be an everywhere positive and continuous function in a domain* Ω *and let* $d_{\mathbf{w}}(\zeta,\eta)$ *be the* $\mathbf{w}$*-distance in* Ω*. Then we have*

$$\sup_{\zeta\in\Omega}\mathbf{w}(\zeta)\|Df(\zeta)\| \quad = \sup_{\zeta,\,\eta\in\Omega,\,\zeta\neq\eta}\frac{|f(\zeta)-f(\eta)|}{d_{\mathbf{w}}(\zeta,\eta)}$$

for any continuously differentiable mappings $f:\Omega\to\mathbf{R}^n$*.*

For $\zeta\in\Omega$ and $\eta\in\Omega$ let

$$\mathbf{W}(\zeta,\eta)=\begin{cases}\mathbf{w}(\zeta), & \text{if } \zeta=\eta,\\ |\zeta-\eta|/d_{\mathbf{w}}(\zeta,\eta), & \text{if } \zeta\neq\eta.\end{cases}$$

It is enough to show that $\mathbf{W}(\zeta,\eta)$ is admissible for $\mathbf{w}(\zeta)$. It is clear that $\mathbf{W}(\zeta,\eta)$ is symmetric. The (W_4)-condition for $\mathbf{W}(\zeta,\eta)$ is obviously satisfied, and here it is optimal in some sense. Therefore, we have only to check if $\mathbf{W}(\zeta,\eta)$ satisfies the (W_3)-condition:

$$\liminf_{\eta\to\zeta}\mathbf{W}(\zeta,\eta)\geq\mathbf{W}(\zeta,\zeta).$$

This means that we need to show that

$$\liminf_{\eta\to\zeta} \frac{|\zeta-\eta|}{d_{\mathbf{w}}(\zeta,\eta)} \geq \mathbf{w}(\zeta).$$

If we invert both sides, we obtain that we have to prove

$$\limsup_{\eta\to\zeta} \frac{d_{\mathbf{w}}(\zeta,\eta)}{|\zeta-\eta|} \leq \frac{1}{\mathbf{w}(\zeta)}.$$

for every $\zeta \in \Omega$.
Since this is a local question, we may assume that η is in a convex neighborhood of ζ. Let γ be among piecewise smooth curves in Ω connecting ζ and η. We have

$$\begin{aligned}\limsup_{\eta\to\zeta} \frac{1}{|\zeta-\eta|} \inf_{\gamma} \int_{\gamma} \frac{|d\omega|}{\mathbf{w}(\omega)} &\leq \limsup_{\eta\to\zeta} \frac{1}{|\zeta-\eta|} \int_{[\zeta,\eta]} \frac{|d\omega|}{\mathbf{w}(\omega)} \\ &= \lim_{\eta\to\zeta} \frac{1}{|\zeta-\eta|} \int_{[\zeta,\eta]} \frac{|d\omega|}{\mathbf{w}(\omega)} = \frac{1}{\mathbf{w}(\zeta)},\end{aligned}$$

which we wanted to prove. The equalities above follow because of continuity of the function $\mathbf{w}(\zeta)$.

A variant of this corollary is obtain in

K. Zhu, *Distances and Banach spaces of holomorphic functions on complex domains*, J. London Math. Soc. **49** (1994), 163–182

(see Theorem 1 there for analytic functions).

As a special case of the above corollary, we have the following one (certainly very well known for analytic Bloch functions in the unit disc).

Corollary

A continuously differentiable mapping $f : \mathbf{B}^m \to \mathbf{R}^n$ is a Bloch mapping (i.e., $f \in \mathcal{B}$) if and only if it is a Lipschitz mapping with respect to the Euclidean and hyperbolic distance in $\mathbf{R}^n$ and $\mathbf{B}^m$. In other words, for the mapping f, there holds

$$|f(\zeta) - f(\eta)| \leq C\rho(\zeta, \eta)$$

for a constant C, if and only if $f \in \mathcal{B}$. Moreover, the optimal constant C is

$$C = \sup\{(1 - |\zeta|^2)\|Df(\zeta)\| : \zeta \in \mathbf{B}^m\}$$

(for a given $f \in \mathcal{B}$)